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# Sign-changing solutions and $W$ -solutions to singular Dirichlet boundary value problems

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# SIGN-CHANGING SOLUTIONS AND $W$ -SOLUTIONS TO SINGULAR DIRICHLET BOUNDARY VALUE PROBLEMS

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**Abstract.** The singular Dirichlet problem  $(r(x)x')' = q(t)f(t, x)$ ,  $x(0) = x(T) = 0$ ,  $\lambda \max\{x(t) : 0 \leq t \leq T\} = -\min\{x(t) : 0 \leq t \leq T\}$  is considered. Here  $f$  is singular at the point  $x = 0$  of the phase variable  $x$  and  $\lambda$  is a positive parameter. The notions of a solution and a  $w$ -solution of the above problem changing its sign exactly once on  $(0, T)$  are introduced. Effective conditions for the existence and multiplicity results are presented. Next, the notion of an exceptional  $n$ -sign-changing  $w$ -solution of our problem with  $\lambda = 1$  is given and for such solutions existence and multiplicity results are proved.

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## 1. Introduction

Consider the problem

$$(r(x(t))x'(t))' = q(t)f(t, x(t)), \quad (1.1)$$

$$x(0) = 0, \quad x(T) = 0, \quad (1.2)$$

$$\lambda \max\{x(t) : 0 \leq t \leq T\} = -\min\{x(t) : 0 \leq t \leq T\}, \quad (1.3)$$

where  $T$  is a positive number,  $\lambda$  is a positive parameter and  $f$  is singular at the point  $x = 0$  of the phase variable  $x$  in the following sense

$$\lim_{x \rightarrow 0^-} f(t, x) = -\infty, \quad \lim_{x \rightarrow 0^+} f(t, x) = \infty \quad \text{for } t \in [0, T]. \quad (1.4)$$

**Definition 1.1.** We say that  $x \in C^1([0, T])$  is a *solution of problem* (1.1) – (1.3) if  $x$  has precisely one zero  $t_0$  on  $(0, T)$ ,  $r(x)x' \in C^1((0, T) \setminus \{t_0\})$ , (1.1) is satisfied for  $t \in (0, T) \setminus \{t_0\}$ ,  $x$  fulfils (1.2) and there exists  $\lambda_0 \in (0, \infty)$  such that (1.3) holds with  $\lambda = \lambda_0$ .

Besides a solution of problem (1.1)-(1.3), we introduce in accordance with [15] the notion of a  $w$ -solution of problem (1.1)-(1.3).

**Definition 1.2.** Let  $\lambda \in (0, \infty)$ . A function  $x \in C^0([0, T])$  is called a *w-solution* of problem (1.1) – (1.3) if  $x$  has precisely one zero  $t_0 \in (0, T)$ ,  $x \in C^1([0, T] \setminus \{t_0\})$ , there exist finite  $\lim_{t \rightarrow t_0^-} x'(t)$  and  $\lim_{t \rightarrow t_0^+} x'(t)$ ,  $r(x)x' \in C^1((0, T) \setminus \{t_0\})$ ,  $x$  fulfils (1.2), (1.3), and (1.1) holds on  $(0, T) \setminus \{t_0\}$ .

We note that in contrast to a solution  $x$  of problem (1.1)-(1.3) which belongs to the class  $C^1([0, T])$  and satisfies (1.3) with a suitable value of  $\lambda$ , a *w-solution*  $x$  of problem (1.1)-(1.3) is continuous on  $[0, T]$ , has continuous derivative on  $[0, t_0) \cup (t_0, T]$  where  $t_0$  is the unique zero of  $x$  in  $(0, T)$  and (1.3) holds with a given value of  $\lambda$ . Naturally, any solution of problem (1.1)-(1.3) is also a *w-solution* of this problem.

In the paper we will use the following assumptions:

- (H<sub>1</sub>)  $r \in C^0(\mathbb{R})$ ,  $r(x) \geq r_0 > 0$  for  $x \in \mathbb{R}$ ;
- (H<sub>2</sub>)  $q \in C^0((0, T))$ ,  $q(t) < 0$  for  $t \in (0, T)$  and  $Q = \sup\{|q(t)| : 0 \leq t \leq T\} < \infty$ ;
- (H<sub>3</sub>)  $f \in C^0([0, T] \times D)$ , where  $D = (-\infty, 0) \cup (0, \infty)$ ,  $f(t, \cdot)$  is nonincreasing on  $D$  for  $t \in [0, T]$  and

$$0 < f(t, x) \operatorname{sign} x \leq g(x) \quad \text{for } (t, x) \in [0, T] \times D,$$

where  $g \in C^0(D)$  and

$$\int_0^0 g(s) ds < \infty, \quad \int_0^0 g(s) ds < \infty;$$

- (H<sub>4</sub>) for each  $(t_0, x_0, x_1) \in (0, T) \times D \times \mathbb{R}$ , there exists a unique solution  $x$  of (1.1) satisfying the initial conditions  $x(t_0) = x_0$ ,  $x'(t_0) = x_1$  defined in a neighbourhood of  $t = t_0$ .

**Remark 1.3.** If  $f$  satisfies (H<sub>3</sub>) then for each  $M > 0$  there exists a positive function  $k_M \in C^0([0, T])$  such that

$$0 < k_M(t) \leq f(t, x) \operatorname{sign} x \leq g(x) \quad \text{on } (t, x) \in [0, T] \times ([-M, 0) \cup (0, M]).$$

Next under the assumption that  $f$  is a locally Lipschitz function on  $(0, T) \times D$ , assumption (H<sub>4</sub>) is satisfied.

In many papers (see, e.g., [1]–[13], [16]–[22] and references therein) only positive (negative) solutions on  $(0, T)$  of the Dirichlet boundary value problems with the singularity at the point  $x = 0$  of the phase variable  $x$  in nonlinearities of considered second-order differential equations have been studied. Solutions were considered either in the class  $C^0([0, T]) \cap C^2((0, T))$  ([1]–[3], [7], [11], [12], [18], [19]) or  $C^1([0, T]) \cap C^2((0, T))$  ([4]–[6], [12], [13], [16]–[19], [22]) or  $C^0([0, T]) \cap AC_{\text{loc}}^1((0, T))$  ([8]–[10], [20], [21]). Here  $AC_{\text{loc}}^1((0, T))$  denotes the set of functions having absolutely continuous first derivatives on any compact subintervals of  $(0, T)$ . The nonlinearities of equations are usually nonpositive ([1], [2], [6]–[8], [11], [12], [16]–[20], [22]), but in [3]–[5], [9], [10], [13] and [21] this assumption is overcome.

For the first time in [14] solutions of singular Dirichlet boundary value problems changing their signs exactly once on  $(0, T)$  were considered. Here differential equations of the form

$$(r(x(t))x'(t))' = \mu q(t)f(t, x(t)) \quad (1.5)$$

together with the condition

$$\max\{x(t) : 0 \leq t \leq T\} \min\{x(t) : 0 \leq t \leq T\} < 0, \quad (1.6)$$

were studied where  $\mu$  is a positive parameter and  $f$  is singular at the point  $x = 0$  of the phase variable  $x$ . A function  $x \in C^1([0, T])$  is called a solution of problem (1.5), (1.2), (1.6) if  $x$  has precisely one zero  $t_0$  on  $(0, T)$ ,  $r(x)x' \in C^1((0, T) \setminus \{t_0\})$ ,  $x$  fulfils (1.2) and (1.6) and there exists  $\mu_0 > 0$  such that (1.5) with  $\mu = \mu_0$  is satisfied for  $t \in (0, T) \setminus \{t_0\}$ . In [14] under assumptions  $(H_1) - (H_3)$  it is proved among others that for each  $A \in (0, \infty)$  there exists a solution  $x$  of problem (1.5), (1.2), (1.6) such that  $\max\{x(t) : 0 \leq t \leq T\} = A$ . We see that any solution of problem (1.5), (1.2), (1.6) depends on a value of the parameter  $\mu$  in equation (1.5) unlike our definition of a solution of problem (1.1)-(1.3) depending on a value of the parameter  $\lambda$  appearing in condition (1.3).

A generalization of the notion of a solution of problem (1.5), (1.2), (1.6) was given in [15]. Here  $x \in C^0([0, T])$  is said to be a  $w$ -solution of problem (1.5), (1.2), (1.6) if  $x$  has precisely one zero  $t_0$  in  $(0, T)$ ,  $x \in C^1([0, T] \setminus \{t_0\})$ , there exist finite  $\lim_{t \rightarrow t_0^-} x'(t)$ ,  $\lim_{t \rightarrow t_0^+} x'(t)$ ,  $r(x)x' \in C^1((0, T) \setminus \{t_0\})$ ,  $x$  fulfils (1.2) and (1.6), and finally there exists  $\mu_0 > 0$  such that (1.5) with  $\mu = \mu_0$  is satisfied for  $t \in (0, T) \setminus \{t_0\}$ . It is proved among others that under assumptions  $(H_1) - (H_3)$  for  $A > 0$  and  $t_0 \in (0, T)$  problem (1.5), (1.2), (1.6) has just two  $w$ -solutions vanishing at  $t_0$  and having their maximum values on  $[0, T]$  equal to  $A$ .

This paper is a continuation of [15] and in comparison with (1.5) our equation (1.1) does not depend on the parameter  $\mu$ . By our definitions any solution as well as any  $w$ -solution  $x$  of problem (1.1)-(1.3) have precisely one zero in  $(0, T)$  where they change their signs. Hence any solution and any  $w$ -solution of problem (1.1)-(1.3) 'pass through' the singularity of  $f$  at a point of the interval  $(0, T)$ .

The paper is organized as follows. In Section 2 we define functions  $\Lambda_+$ ,  $\Phi_+$ ,  $\Lambda_-$  and  $\Phi_-$  by (2.6)-(2.9) and present some of their important properties. By these functions we prove existence and uniqueness results for  $w$ -solutions of problem (1.1)-(1.3) in Section 3 (Theorems 3.1 and 3.2). Section 4 is devoted to the study of existence and multiplicity results for solutions of problem (1.1)-(1.3) (Theorem 4.5). In Section 5 we first give the notion of an exceptional  $n$ -sign-changing  $w$ -solution  $x$  of problem (1.1), (1.2), (5.1) which changes its sign exactly  $n$  times on  $[0, T]$  and on each maximum subinterval of  $[0, T]$  where  $x$  keeping its sign the function  $|x|$  has the same maximum value. Existence and multiplicity results for the  $n$ -sign-changing  $w$ -solution of problem (1.1), (1.2), (5.1) are given in Theorem 5.7.

## 2. Lemmas, notation

Let  $0 \leq a < b \leq T$ . In our consideration we will work with the following auxiliary boundary conditions

$$x(a) = x(b) = 0, \quad x(t) > 0 \quad \text{for } t \in (a, b), \quad (2.1)$$

$$x(a) = x(b) = 0, \quad x(t) < 0 \quad \text{for } t \in (a, b) \quad (2.2)$$

and we will use the function  $H : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$H(u) = \int_0^u r(s) ds \quad (2.3)$$

with  $r$  occurring in (1.1) and satisfying assumption  $(H_1)$ . Clearly,  $H \in C^1(\mathbb{R})$  is increasing on  $\mathbb{R}$  and the inverse function to  $H$  denoted by  $H^{-1}$  is increasing on  $\mathbb{R}$ .

We say that  $x$  is a *solution of problem* (1.1), (j),  $j \in \{2.1, 2.2\}$  if  $x \in C^1([a, b])$ ,  $r(x)x' \in C^1((a, b))$ ,  $x$  satisfies the boundary conditions (j) and (1.1) is fulfilled for  $t \in (a, b)$ .

**Remark 2.1.** Let the function  $\tilde{q} : (0, T) \rightarrow (-\infty, 0)$  and  $\tilde{f} : [0, T] \times D \rightarrow \mathbb{R}$  be defined by

$$\tilde{q}(t) = q(T - t), \quad \tilde{f}(t, x) = f(T - t, x).$$

Then

$$0 < \tilde{f}(t, x) \operatorname{sign} x < g(x), \quad (t, x) \in [0, T] \times D$$

and assumptions  $(H_2)$ – $(H_4)$  are satisfied with  $\tilde{q}$  and  $\tilde{f}$  instead of  $q$  and  $f$ . If we consider the differential equation

$$(r(x(t))x'(t))' = \tilde{q}(t)\tilde{f}(t, x(t)), \quad (2.4)$$

we see that a function  $x$  is a solution of problem (1.1), (j) with  $a = 0$ ,  $b = c (< T)$  and  $j \in \{2.1, 2.2\}$  if the function  $\tilde{x}(t) = x(T - t)$ ,  $t \in [T - c, T]$ , is a solution of problem (2.4), (j) with  $a = T - c$ ,  $b = T$ . Conversely, if  $\tilde{x}$  is a solution of problem (2.4), (j) with  $a = T - c (> 0)$ ,  $b = T$  and  $j \in \{2.1, 2.2\}$ , then the function  $x(t) = \tilde{x}(T - t)$ ,  $t \in [0, c]$ , is a solution of problem (1.1), (j) with  $a = 0$ ,  $b = c$ .

**Remark 2.2.** Let  $r^* : \mathbb{R} \rightarrow [r_0, \infty)$ ,  $f_* : [0, T] \times D \rightarrow \mathbb{R}$  and  $g^* : D \rightarrow \mathbb{R}$  be defined by the formulas (see [14])

$$r^*(x) = r(-x), \quad f^*(t, x) = -f(t, -x), \quad g^*(x) = g(-x).$$

Then

$$0 < f^*(t, x) \operatorname{sign} x \leq g^*(x), \quad (t, x) \in [0, T] \times D$$

and assumptions  $(H_1)$ – $(H_4)$  are satisfied with  $r^*$ ,  $f^*$  and  $g^*$  instead of  $r$ ,  $f$  and  $g$ . It is easily seen that a function  $x$  is a solution of problem (1.1), (j),  $j \in \{2.1, 2.2\}$ , if and only if  $x^* = -x$  on  $[a, b]$  is a solution of problem (2.5), (j), where

$$(r^*(x(t))x'(t))' = q(t)f^*(t, x(t)). \quad (2.5)$$

**Lemma 2.3.** Let assumptions  $(H_1) - (H_3)$  be satisfied. Then for each  $a, b \in [0, T]$ ,  $a < b$ , there exists a unique solution of problem (1.1), (2.1).

**Proof.** The assertion of our lemma follows from Theorem 2.1 in [14] with  $\mu = 1$ .  $\square$

**Corollary 2.4.** Under assumptions of Lemma 2.3, for each  $a, b \in [0, T]$ ,  $a < b$ , there exists a unique solution of problem (1.1), (2.2).

**Proof.** Fix  $0 \leq a < b \leq T$ . Since assumptions  $(H_1) - (H_3)$  are satisfied with the functions  $r^*$ ,  $f^*$  and  $g^*$  defined in Remark 2.2 instead of  $r$ ,  $f$  and  $g$ , problem (2.5), (2.1) has a unique solution  $\tilde{x}$  by Lemma 2.3. Now the function  $x = -\tilde{x}$  on  $[a, b]$  is the unique solution of problem (1.1), (2.2).  $\square$

For each  $\alpha \in (0, T]$  and  $\beta \in [0, T)$ , we denote throughout this paper by  $u_\alpha$  and  $v_\beta$  the unique solution of problem (1.1), (2.1) with  $a = 0$ ,  $b = \alpha$  and  $a = \beta$ ,  $b = T$ , respectively. Next by  $\bar{u}_\alpha$  and  $\bar{v}_\beta$  we denote the unique solution of problem (1.1), (2.2) with  $a = 0$ ,  $b = \alpha$  and  $a = \beta$ ,  $b = T$ , respectively. The existence and uniqueness of  $u_\alpha$ ,  $v_\beta$  and  $\bar{u}_\alpha$ ,  $\bar{v}_\beta$  follow from Lemma 2.3 and Corollary 2.4, respectively.

**Lemma 2.5.** (Lemma 2.7 in [14].) Let assumptions  $(H_1) - (H_3)$  be satisfied and let  $0 < \alpha_1 < \alpha_2 \leq T$ . Then

$$u_{\alpha_1}(t) \leq u_{\alpha_2}(t) \quad \text{for } t \in [0, \alpha_1].$$

By the solutions  $u_\alpha$ ,  $v_\beta$ ,  $\bar{u}_\alpha$  and  $\bar{v}_\beta$  define the functions  $\Lambda_+ : (0, T] \rightarrow (0, \infty)$ ,  $\Phi_+ : [0, T) \rightarrow (0, \infty)$ ,  $\Lambda_- : (0, T] \rightarrow (-\infty, 0)$ ,  $\Phi_- : [0, T) \rightarrow (-\infty, 0)$  by the formulas

$$\Lambda_+(\alpha) = \max\{u_\alpha(t) : 0 \leq t \leq \alpha\}, \quad (2.6)$$

$$\Phi_+(\beta) = \max\{v_\beta(t) : \beta \leq t \leq T\}, \quad (2.7)$$

$$\Lambda_-(\alpha) = \min\{\bar{u}_\alpha(t) : 0 \leq t \leq \alpha\} \quad (2.8)$$

and

$$\Phi_-(\beta) = \min\{\bar{v}_\beta(t) : \beta \leq t \leq T\}. \quad (2.9)$$

Properties of the functions  $\Lambda_+$ ,  $\Phi_+$ ,  $\Lambda_-$  and  $\Phi_-$  are presented in the following lemmas.

**Lemma 2.6.** Let assumptions  $(H_1) - (H_3)$  be satisfied. Then  $\Lambda_+$  is continuous nondecreasing on  $(0, T]$  and

$$\lim_{\alpha \rightarrow 0^+} \Lambda_+(\alpha) = 0.$$

**Proof.** As a direct consequence of Lemma 2.5 we get that  $\Lambda_+$  is nondecreasing on  $(0, T]$ . Suppose that  $\Lambda_+$  is discontinuous on the right at a point  $\alpha_0 \in (0, T)$ , i.e. there is a decreasing sequence  $\{\alpha_n\} \subset (\alpha_0, T)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = \alpha_0$  and

$$\lim_{n \rightarrow \infty} \Lambda_+(\alpha_n) > \Lambda_+(\alpha_0). \quad (2.10)$$

Consider the sequence  $\{u_{\alpha_n}\}$ . Since  $(r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' = q(t)f(t, u'_{\alpha_n}(t)) < 0$  for  $t \in (0, \alpha_n)$ ,  $r(u_{\alpha_n})u'_{\alpha_n}$  is decreasing on  $[0, \alpha_n]$  and therefore there exists a (unique)  $\xi_n \in (0, \alpha_n)$  such that  $u'_{\alpha_n} > 0$  on  $[0, \xi_n]$ ,  $u'_{\alpha_n} < 0$  on  $(\xi_n, \alpha_n]$  and  $u'_{\alpha_n}(\xi_n) = 0$ . Integrating the inequalities

$$\begin{aligned} & (r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' r(u_{\alpha_n}(t))u'_{\alpha_n}(t) \\ & \geq -Qg(u_{\alpha_n}(t))r(u_{\alpha_n}(t))u'_{\alpha_n}(t), \quad t \in (0, \xi_n) \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} & (r(u_{\alpha_n}(t))u'_{\alpha_n}(t))' r(u_{\alpha_n}(t))u'_{\alpha_n}(t) \\ & \leq -Qg(u_{\alpha_n}(t))r(u_{\alpha_n}(t))u'_{\alpha_n}(t), \quad t \in (\xi_n, \alpha_n) \end{aligned} \quad (2.12)$$

over  $[0, \xi_n]$  and  $[\xi_n, \alpha_n]$ , we obtain

$$(r(0)u'_{\alpha_n}(0))^2 \leq 2Q \int_0^{\Lambda_+(\alpha_n)} g(s)r(s) ds \leq 2Q \int_0^{\Lambda_+(\alpha_1)} g(s)r(s) ds$$

and

$$(r(0)u'_{\alpha_n}(\alpha_n))^2 \leq 2Q \int_0^{\Lambda_+(\alpha_n)} g(s)r(s) ds \leq 2Q \int_0^{\Lambda_+(\alpha_1)} g(s)r(s) ds,$$

respectively. Hence

$$\begin{aligned} & |r(u_{\alpha_n}(t))u'_{\alpha_n}(t)| \leq r(0) \max\{|u'_{\alpha_n}(0)|, |u'_{\alpha_n}(\alpha_n)|\} \\ & \leq \sqrt{2Q \int_0^{\Lambda_+(\alpha_1)} g(s)r(s) ds}, \quad t \in [0, \alpha_n], \quad n \in \mathbb{N} \end{aligned} \quad (2.13)$$

and

$$|u'_{\alpha_n}(t)| \leq \frac{1}{r_0} \sqrt{2Q \int_0^{\Lambda_+(\alpha_1)} g(s)r(s) ds} \quad \text{for } t \in [0, \alpha_n], \quad n \in \mathbb{N}. \quad (2.14)$$

In addition, by Lemma 2.5,

$$u_{\alpha_0}(t) \leq u_{\alpha_n}(t) \leq u_{\alpha_{n+1}}(t) \quad \text{for } t \in [0, \alpha_0], \quad n \in \mathbb{N}. \quad (2.15)$$

From (2.14) and (2.15) we deduce that  $\{u_{\alpha_n}(t)\}$  is uniformly convergent on  $[0, \alpha_0]$  and let  $\lim_{n \rightarrow \infty} u_{\alpha_n}(t) = u(t)$ ,  $t \in [0, \alpha_0]$ . Then  $u \in C^0([0, \alpha_0])$ ,  $u(0) = 0$ ,  $u(t) \geq u_{\alpha_0}(t) > 0$  for  $t \in (0, \alpha_0)$ . Moreover,  $u(\alpha_0) = 0$ , since in the case that  $u(\alpha_0) > 0$  it may be concluded from

$$u(\alpha_0) \leq u_{\alpha_n}(\alpha_0) = u_{\alpha_n}(\alpha_0) - u_{\alpha_n}(\alpha_n) = u'_{\alpha_n}(\eta_n)(\alpha_0 - \alpha_n),$$

where  $\eta_n \in (\alpha_0, \alpha_n)$  that

$$\lim_{n \rightarrow \infty} u'_{\alpha_n}(\eta_n) \leq \lim_{n \rightarrow \infty} \frac{u(\alpha_0)}{\alpha_0 - \alpha_n} = -\infty,$$

contrary to (2.14). As

$$0 < f(t, u_{\alpha_{n+1}}(t)) \leq f(t, u_{\alpha_n}(t)), \quad \lim_{n \rightarrow \infty} f(t, u_{\alpha_n}(t)) = f(t, u(t)), \quad t \in (0, \alpha_0)$$

and (see (2.13))

$$\begin{aligned} 0 &> \int_0^{\alpha_0} q(t) f(t, u_{\alpha_n}(t)) dt = r(u_{\alpha_n}(\alpha_0)) u'_{\alpha_n}(\alpha_0) - r(0) u'_{\alpha_n}(0) \\ &\geq -2 \sqrt{2Q \int_0^{\Lambda_+(\alpha_1)} g(s) r(s) ds} \end{aligned}$$

for  $n \in \mathbb{N}$ , Fatou's and Levi's theorems give  $q(\cdot) f(\cdot, u(\cdot)) \in L_1([0, \alpha_0])$  and

$$\lim_{n \rightarrow \infty} \int_0^t q(s) f(s, u_{\alpha_n}(s)) ds = \int_0^t q(s) f(s, u(s)) ds, \quad t \in [0, \alpha_0].$$

By (2.14),  $\{u'_{\alpha_n}(0)\}$  is bounded and we may assume that it is convergent. Let  $\lim_{n \rightarrow \infty} u'_{\alpha_n}(0) = A$ . Letting  $n \rightarrow \infty$  in

$$H(u_{\alpha_n}(t)) = r(0) u'_{\alpha_n}(0) t + \int_0^t \int_0^s q(v) f(v, u_{\alpha_n}(v)) dv ds \quad \text{for } t \in [0, \alpha_0], \quad (2.16)$$

where  $H$  is given by (2.3), we get

$$H(u(t)) = r(0) A t + \int_0^t \int_0^s q(v) f(v, u(v)) dv ds, \quad t \in [0, \alpha_0].$$

Then

$$u(t) = H^{-1} \left( r(0) A t + \int_0^t \int_0^s q(v) f(v, u(v)) dv ds \right),$$

and so  $u \in C^1([0, \alpha_0])$ . Now from  $r(u(t)) u'(t) = r(0) A + \int_0^t q(s) f(s, u(s)) ds$ ,  $t \in [0, \alpha_0]$ , and the above proved properties of  $u$ , we see that  $u$  is a solution of problem (1.1), (2.1) with  $a = 0$  and  $b = \alpha_0$ , and consequently  $u = u_{\alpha_0}$  by Lemma 2.3. We have proved that  $\lim_{n \rightarrow \infty} u_{\alpha_n}(t) = u_{\alpha_0}(t)$  uniformly on  $[0, \alpha_0]$ , which implies  $\lim_{n \rightarrow \infty} \Lambda_+(\alpha_n) = \Lambda_+(\alpha_0)$ , contrary to (2.10). Hence  $\Lambda_+$  is continuous on the right on  $(0, T)$ .

Assume now that  $\Lambda_+$  is discontinuous on the left at a point  $\alpha_0 \in (0, T]$ , i.e., there is an increasing sequence  $\{\alpha_n\} \subset (0, \alpha_0)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = \alpha_0$  and

$$\lim_{n \rightarrow \infty} \Lambda_+(\alpha_n) < \Lambda_+(\alpha_0). \quad (2.17)$$

Then

$$u_{\alpha_n}(t) \leq u_{\alpha_{n+1}}(t) \leq u_{\alpha_0}(t) \quad \text{for } t \in [0, \alpha_n], \quad n \in \mathbb{N} \quad (2.18)$$



by Lemma 2.5 and as above it can be verified that

$$\begin{aligned} |r(u_{\alpha_n}(t))u'_{\alpha_n}(t)| &\leq \sqrt{2Q \int_0^{\Lambda_+(\alpha_0)} g(s)r(s) ds}, \\ |u'_{\alpha_n}(t)| &\leq \frac{1}{r_0} \sqrt{2Q \int_0^{\Lambda_+(\alpha_0)} g(s)r(s) ds} \end{aligned} \quad (2.19)$$

and

$$0 \geq \int_0^t q(s)f(s, u_{\alpha_n}(s)) ds \geq -2\sqrt{2Q \int_0^{\Lambda_+(\alpha_0)} g(s)r(s) ds} \quad (2.20)$$

for  $t \in [0, \alpha_n]$  and  $n \in \mathbb{N}$ . By (2.18) and (2.19),  $\{u_{\alpha_n}(t)\}$  is locally uniformly convergent on  $[0, \alpha_0]$  and let  $\lim_{n \rightarrow \infty} u_{\alpha_n}(t) = u(t)$ ,  $t \in [0, \alpha_0]$ . Then  $u \in C^0([0, \alpha_0])$ ,  $u(0) = 0$ ,  $0 < u(t) \leq u_{\alpha_0}(t)$  for  $t \in [0, \alpha_0]$  and  $\lim_{t \rightarrow \alpha_0^-} u(t) = 0$ . From the inequalities  $f(t, u_{\alpha_n}(t)) \geq f(t, u_{\alpha_{n+1}}(t)) \geq f(t, u_{\alpha_0}(t)) \geq 0$ , (2.20),  $\lim_{n \rightarrow \infty} f(t, u_{\alpha_n}(t)) = f(t, u(t))$  for  $t \in (0, \alpha_0)$  and Fatou's theorem we obtain  $q(\cdot)f(\cdot, u(\cdot)) \in L_1([0, \alpha_0])$ . Define  $u^* \in C^0([0, \alpha_0])$  by

$$u^*(t) = \begin{cases} u(t) & \text{for } t \in [0, \alpha_0) \\ 0 & \text{for } t = \alpha_0. \end{cases}$$

Without violating generality, we can assume that  $\{u'_{\alpha_n}(0)\}$  is convergent and let  $\lim_{n \rightarrow \infty} u'_{\alpha_n}(0) = B$ . Taking the limit as  $n \rightarrow \infty$  in (2.16) which now holds on  $[0, \alpha_n]$ , we obtain

$$H(u^*(t)) = r(0)Bt + \int_0^t \int_0^s q(v)f(v, u^*(v)) dv ds \quad \text{for } t \in [0, \alpha_0].$$

Then  $u^* \in C^1([0, \alpha_0])$  and  $u^*$  is a solution of problem (1.1), (2.1) with  $a = 0$  and  $b = \alpha_0$ . Hence  $u^* = u_{\alpha_0}$  and from  $\lim_{n \rightarrow \infty} u_{\alpha_n}(t) = u_{\alpha_0}(t)$  locally uniformly on  $[0, \alpha_0]$  we deduce that  $\lim_{n \rightarrow \infty} \Lambda_+(\alpha_n) = \Lambda_+(\alpha_0)$ , contrary to (2.17). It follows that  $\Lambda_+$  is continuous on the left on  $(0, T]$ . Consequently,  $\Lambda_+$  is continuous on  $(0, T]$ .

Finally, assume that  $\lim_{\alpha \rightarrow 0^+} \Lambda_+(\alpha) = \mu > 0$ . Let  $\{\alpha_n\} \subset (0, T)$  be a decreasing sequence and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ . Let  $\Lambda_+(\alpha_n) = u_{\alpha_n}(\xi_n)$  with a  $\xi_n \in (0, \alpha_n)$ . Then  $u_{\alpha_n}(\xi_n) \geq \mu$  and from  $\mu \leq u_{\alpha_n}(\xi_n) = u_{\alpha_n}(\xi_n) - u_{\alpha_n}(0) = u'_{\alpha_n}(\tau_n)\xi_n$ , where  $\tau_n \in (0, \xi_n)$ , we have  $u'_{\alpha_n}(\tau_n) \geq \mu/\xi_n$  for  $n \in \mathbb{N}$ . Therefore  $\lim_{n \rightarrow \infty} u'_{\alpha_n}(\tau_n) \geq \lim_{n \rightarrow \infty} \mu/\xi_n = \infty$ , contrary to (2.14). Hence  $\lim_{\alpha \rightarrow 0^+} \Lambda_+(\alpha) = 0$ .  $\square$

**Lemma 2.7.** Let assumptions  $(H_1) - (H_3)$  be satisfied. Then  $\Phi_+$  is continuous nonincreasing on  $[0, T)$  and

$$\lim_{\beta \rightarrow T^-} \Phi_+(\beta) = 0.$$

**Proof.** By Remark 2.1, for each  $\gamma \in (0, T]$  the function  $\tilde{u}_\gamma(t) = v_\gamma(T - t)$ ,  $t \in [0, T - \gamma]$ , is a (unique) solution of problem (2.4), (2.1) with  $a = 0$  and  $b = T - \gamma$ . Set  $\tilde{\Lambda}_+(\gamma) = \max\{\tilde{u}_\gamma(t) : 0 \leq t \leq \gamma\}$  for  $\gamma \in (0, T]$ . Applying Lemma 2.6 to equation (2.4), we see that  $\tilde{\Lambda}_+$  is continuous and nondecreasing on  $(0, T]$  and  $\lim_{\gamma \rightarrow 0^+} \tilde{\Lambda}_+(\gamma) =$

0. The assertion of our lemma now follows from the equality  $\Phi_+(\beta) = \tilde{\Lambda}_+(T - \beta)$  for  $\beta \in [0, T)$ .  $\square$

**Lemma 2.8.** Let assumptions  $(H_1) - (H_3)$  be satisfied. Then  $\Lambda_-$  is continuous nonincreasing on  $(0, T]$ ,  $\Phi_-$  is continuous nondecreasing on  $[0, T)$  and

$$\lim_{\alpha \rightarrow 0^+} \Lambda_-(\alpha) = \lim_{\beta \rightarrow T^-} \Phi_-(\beta) = 0.$$

**Proof.** Let  $\tilde{\Lambda}_+$  and  $\tilde{\Phi}_+$  be associated to problem (2.5), (2.1) analogously as  $\Lambda_+$  and  $\Phi_+$  are to problem (1.1), (2.1). Then  $\tilde{\Lambda}_+$  is continuous and nondecreasing on  $(0, T]$ ,  $\tilde{\Phi}_+$  is continuous and nonincreasing on  $[0, T)$  and  $\lim_{\alpha \rightarrow 0^+} \tilde{\Lambda}_+(\alpha) = \lim_{\beta \rightarrow T^-} \tilde{\Phi}_+(\beta) = 0$  by Lemmas 2.6 and 2.7. The assertions of the lemma follow immediately from the equalities  $\Lambda_- = -\tilde{\Lambda}_+$  on  $(0, T]$  and  $\Phi_- = -\tilde{\Phi}_+$  on  $[0, T)$  which we get applying Remark 2.2.  $\square$

**Lemma 2.9.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then for each  $\alpha_1, \alpha_2 \in (0, T]$ ,  $\alpha_1 < \alpha_2$ , the inequality

$$u_{\alpha_1}(t) < u_{\alpha_2}(t) \quad \text{for } t \in (0, \alpha_1] \quad (2.21)$$

holds.

**Proof.** Fix  $0 < \alpha_1 < \alpha_2 \leq T$ . Then  $0 = u_{\alpha_1}(\alpha_1) < u_{\alpha_2}(\alpha_1)$  and, by Lemma 2.5,  $u_{\alpha_1}(t) \leq u_{\alpha_2}(t)$  for  $t \in (0, \alpha_1]$ . If  $u_{\alpha_1}(\xi) = u_{\alpha_2}(\xi)$  for some  $\xi \in (0, \alpha_1)$ , then  $u'_{\alpha_1}(\xi) = u'_{\alpha_2}(\xi)$ , and consequently  $u_{\alpha_1} = u_{\alpha_2}$  in a neighbourhood of  $t = \xi$  by assumption  $(H_4)$ . Repeated application of this result enables us to prove that  $u_{\alpha_1} = u_{\alpha_2}$  on  $[0, \alpha_1)$ , which is impossible. Hence (2.21) holds.  $\square$

**Lemma 2.10.** Under assumptions  $(H_1) - (H_4)$ ,  $\Lambda_+$  is increasing on  $(0, T]$ ,  $\Phi_+$  is decreasing on  $[0, T)$ ,  $\Lambda_-$  is decreasing on  $(0, T]$  and  $\Phi_-$  is increasing on  $[0, T)$ .

**Proof.** By Lemma 2.9, for each  $0 < \alpha_1 < \alpha_2 \leq T$ , inequality (2.21) holds and from the definition of  $\Lambda_+$  we have  $\Lambda_+(\alpha_1) < \Lambda_+(\alpha_2)$ . Hence  $\Lambda_+$  is increasing on  $(0, T]$ . The other three assertions of the lemma can be verified from strict inequalities between solutions  $v_{\alpha_1}, v_{\alpha_2}; \bar{u}_{\alpha_1}, \bar{u}_{\alpha_2}$  and  $\bar{v}_{\alpha_1}, \bar{v}_{\alpha_2}$  with different  $\alpha_1$  and  $\alpha_2$ .  $\square$

### 3. Existence results for $w$ -solutions of problem (1.1)-(1.3)

**Theorem 3.1.** Let assumptions  $(H_1) - (H_3)$  be satisfied. Then for each  $\lambda \in (0, \infty)$  there exist at least two  $w$ -solutions of problem (1.1)-(1.3).

**Proof.** Fix  $\lambda \in (0, \infty)$ . By Lemmas 2.6 and 2.8, the function  $\lambda\Lambda_+ + \Phi_-$  is continuous and nondecreasing on  $(0, T)$  and  $\lim_{\alpha \rightarrow 0^+} (\lambda\Lambda_+(\alpha) + \Phi_-(\alpha)) = \Phi_-(0) < 0$ ,

$\lim_{\alpha \rightarrow T^-} (\lambda \Lambda_+(\alpha) + \Phi_-(\alpha)) = \lambda \Lambda_+(T) > 0$ . Hence the equation  $\lambda \Lambda_+(\alpha) + \Phi_-(\alpha) = 0$  has at least one solution  $\alpha_1 \in (0, T)$ . Setting

$$x_1(t) = \begin{cases} u_{\alpha_1}(t) & \text{for } t \in [0, \alpha_1] \\ \bar{v}_{\alpha_1}(t) & \text{for } t \in (\alpha_1, T], \end{cases} \quad (3.1)$$

$x_1$  is a  $w$ -solution of problem (1.1)-(1.3). Analogously, the equation  $\lambda \Phi_+(\alpha) + \Lambda_-(\alpha) = 0$  has at least one solution  $\alpha_2 \in (0, T)$  since  $\lambda \Phi_+ + \Lambda_-$  is continuous and nonincreasing on  $(0, T)$  and  $\lim_{\alpha \rightarrow 0^+} (\lambda \Phi_+(\alpha) + \Lambda_-(\alpha)) = \lambda \Phi_+(0) > 0$ ,  $\lim_{\alpha \rightarrow T^-} (\lambda \Phi_+(\alpha) + \Lambda_-(\alpha)) = \Lambda_-(T) < 0$  by Lemmas 2.7 and 2.8. Then setting

$$x_2(t) = \begin{cases} \bar{u}_{\alpha_2}(t) & \text{for } t \in [0, \alpha_2] \\ v_{\alpha_2}(t) & \text{for } t \in (\alpha_2, T], \end{cases} \quad (3.2)$$

$x_2$  is the second  $w$ -solution of problem (1.1)-(1.3). From  $x_1 > 0$  on  $(0, \alpha_1)$  and  $x_2 < 0$  on  $(0, \alpha_2)$  we see that  $x_1 \neq x_2$ .  $\square$

**Theorem 3.2.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then for each  $\lambda \in (0, \infty)$  there exist precisely two  $w$ -solutions of problem (1.1)-(1.3).

**Proof.** Fix  $\lambda \in (0, \infty)$ . It follows from Lemma 2.10 and the properties of the functions  $\Lambda_+$ ,  $\Phi_+$ ,  $\Lambda_-$  and  $\Phi_-$  given in Lemmas 2.6–2.8 that the equations  $\lambda \Lambda_+(\alpha) + \Phi_-(\alpha) = 0$  and  $\lambda \Phi_+(\alpha) + \Lambda_-(\alpha) = 0$  have in  $(0, T)$  the unique solutions  $\alpha_1$  and  $\alpha_2$ , respectively. Now  $x_1$  and  $x_2$  defined by (3.1) and (3.2) are unique  $w$ -solutions of problem (1.1)-(1.3).  $\square$

#### 4. Existence results for solutions of problem (1.1)–(1.3)

Let assumptions  $(H_1) - (H_4)$  be satisfied. By Theorem 3.2, for each  $\lambda \in (0, \infty)$  there exist precisely two  $w$ -solutions  $x_1(t; \lambda)$  and  $x_2(t; \lambda)$  of problem (1.1)-(1.3). If  $c_\lambda$  is the (unique) solution of the equation  $\lambda \Lambda_+(c) + \Phi_-(c) = 0$  and  $\alpha_\lambda$  is the (unique) solution of the equation  $\lambda \Phi_+(\alpha) + \Lambda_-(\alpha) = 0$ , then

$$x_1(t; \lambda) = \begin{cases} u_{c_\lambda}(t) & \text{for } t \in [0, c_\lambda] \\ \bar{v}_{c_\lambda}(t) & \text{for } t \in (c_\lambda, T] \end{cases}$$

and

$$x_2(t; \lambda) = \begin{cases} \bar{u}_{\alpha_\lambda}(t) & \text{for } t \in [0, \alpha_\lambda] \\ v_{\alpha_\lambda}(t) & \text{for } t \in (\alpha_\lambda, T]. \end{cases}$$

Here solutions  $u_\alpha$ ,  $v_\beta$ ,  $\bar{u}_\alpha$  and  $\bar{v}_\beta$  were defined in Section 2. Of course,

$$\lambda \max\{u_{c_\lambda}(t) : 0 \leq t \leq c_\lambda\} = -\min\{\bar{v}_{c_\lambda} : c_\lambda \leq t \leq T\},$$

$$\lambda \max\{v_{\alpha_\lambda}(t) : \alpha_\lambda \leq t \leq T\} = -\min\{\bar{u}_{\alpha_\lambda} : 0 \leq t \leq \alpha_\lambda\}$$

and  $c_\lambda$  (resp.  $\alpha_\lambda$ ) is the (unique) zero of  $x_1(t; \lambda)$  (resp.  $x_2(t; \lambda)$ ) in  $(0, T)$ .

**Lemma 4.1.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then

a)  $c_\lambda$  is continuous and decreasing on  $(0, \infty)$ ,

$$\lim_{\lambda \rightarrow \infty} c_\lambda = 0, \quad \lim_{\lambda \rightarrow 0^+} c_\lambda = T,$$

b)  $u_{c_{\lambda_1}}(t) > u_{c_{\lambda_2}}(t)$  for  $t \in (0, c_{\lambda_2}]$  and  $0 < \lambda_1 < \lambda_2$ .

**Proof.** We know (see the proof of Theorem 3.2) that  $c_\lambda$  is the (unique) solution of the equation  $\lambda\Lambda_+(c) + \Phi_-(c) = 0$ . Hence the equality  $\lambda\Lambda_+(c_\lambda) + \Phi_-(c_\lambda) = 0$  holds for  $\lambda \in (0, \infty)$ . Let  $0 < \lambda_1 < \lambda_2$ . If  $c_{\lambda_1} \leq c_{\lambda_2}$ , then from the properties of the functions  $\Lambda_+$  and  $\Phi_-$  given in Lemmas 2.6–2.10 it follows that  $0 = \lambda_1\Lambda_+(c_{\lambda_1}) + \Phi_-(c_{\lambda_1}) < \lambda_2\Lambda_+(c_{\lambda_2}) + \Phi_-(c_{\lambda_2}) = 0$ , contrary to  $\lambda_2\Lambda_+(c_{\lambda_2}) + \Phi_-(c_{\lambda_2}) = 0$ . Therefore  $c_\lambda$  is decreasing on  $(0, \infty)$ .

Assume that  $c_\lambda$  is discontinuous at a point  $\lambda_0 \in (0, \infty)$ . Then there is a sequence  $\{\lambda_n\} \subset (0, \infty)$ ,  $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$  such that  $\lim_{n \rightarrow \infty} c_{\lambda_n} = \mu_0 \neq c_{\lambda_0}$ . Letting  $n \rightarrow \infty$  in the equalities  $\lambda_n\Lambda_+(c_{\lambda_n}) + \Phi_-(c_{\lambda_n}) = 0$ ,  $n \in \mathbb{N}$ , we get

$$\lambda_0\Lambda_+(\mu_0) + \Phi_-(\mu_0) = 0 \quad (4.1)$$

since  $\Lambda_+$  and  $\Phi_-$  are continuous. But the equation  $\lambda_0\Lambda_+(c) + \Phi_-(c) = 0$  has the unique solution  $c = c_{\lambda_0}$ , contrary to (4.1). Hence  $c_\lambda$  is continuous on  $(0, \infty)$ .

Suppose  $\lim_{\lambda \rightarrow \infty} c_\lambda = \mu > 0$ . Then  $\Lambda_+(c_\lambda) \geq \Lambda_+(\mu) > 0$  and  $\Phi_-(c_\lambda) \leq \Phi_-(\mu) < 0$  for  $\lambda \in (0, \infty)$ , and so  $\lim_{\lambda \rightarrow \infty} (\lambda\Lambda_+(c_\lambda) + \Phi_-(c_\lambda)) = \infty$ , contrary to

$$\lambda\Lambda_+(c_\lambda) + \Phi_-(c_\lambda) = 0 \quad \text{for } \lambda \in (0, \infty). \quad (4.2)$$

Therefore  $\lim_{\lambda \rightarrow \infty} c_\lambda = 0$ . If  $\lim_{\lambda \rightarrow 0^+} c_\lambda = \varrho < T$ , then  $\lim_{\lambda \rightarrow 0^+} \Lambda_+(c_\lambda) = \Lambda_+(\varrho) > 0$ ,  $\lim_{\lambda \rightarrow 0^+} \Phi_-(c_\lambda) = \Phi_-(\varrho) < 0$ , and so  $\lim_{\lambda \rightarrow 0^+} (\lambda\Lambda_+(c_\lambda) + \Phi_-(c_\lambda)) = \Phi_-(\varrho) < 0$ , contrary to (4.2). Hence  $\lim_{\lambda \rightarrow 0^+} c_\lambda = T$ .

Finally, if  $0 < \lambda_1 < \lambda_2$ , then  $c_{\lambda_1} > c_{\lambda_2}$  and  $u_{c_{\lambda_1}}(t) > u_{c_{\lambda_2}}(t)$  for  $t \in (0, c_{\lambda_2}]$  by Lemma 2.9.  $\square$

**Lemma 4.2.** Let assumptions  $(H_1) - (H_4)$  be satisfied and let  $\{\lambda_n\} \subset (0, \infty)$ ,  $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0 > 0$ . Then

$$\lim_{n \rightarrow \infty} u_{c_{\lambda_n}}(t) = u_{c_{\lambda_0}}(t) \quad \text{locally uniformly on } [0, c_{\lambda_0}).$$

**Proof.** First from (2.14) it follows that

$$|u'_{c_{\lambda_n}}(t)| \leq \frac{1}{r_0} \sqrt{2Q \int_0^{\Lambda_+(T)} g(s)r(s) ds} \quad \text{for } t \in [0, c_{\lambda_n}], \quad n \in \mathbb{N}. \quad (4.3)$$

Now from (4.3) and using the fact that for  $\{\lambda_n\}$  decreasing,  $\{c_{\lambda_n}\}$  is increasing and

$$u_{c_{\lambda_{n+1}}}(t) > u_{c_{\lambda_n}}(t), \quad t \in [0, c_{\lambda_n}], \quad n \in \mathbb{N}$$

and for  $\{\lambda_n\}$  increasing,  $\{c_{\lambda_n}\}$  is decreasing and

$$u_{c_{\lambda_{n+1}}}(t) < u_{c_{\lambda_n}}(t), \quad t \in [0, c_{\lambda_{n+1}}], \quad n \in \mathbb{N}$$

(see Lemma 4.1), we deduce the assertion of our lemma.  $\square$

Define the function  $S_+ : (0, \infty) \rightarrow (-\infty, 0]$  by the formula

$$S_+(\lambda) = u'_{c_\lambda}(c_\lambda),$$

where  $u'_{c_\lambda}(c_\lambda)$  denotes the derivative of  $u_{c_\lambda}(t)$  on the left at the point  $t = c_\lambda$ .

**Lemma 4.3.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then  $S_+$  is continuous on  $(0, \infty)$  and

$$\lim_{\lambda \rightarrow \infty} S_+(\lambda) = 0, \quad \limsup_{\lambda \rightarrow 0^+} S_+(\lambda) < 0.$$

**Proof.** Assume, on the contrary, that  $S_+$  is discontinuous at a point  $\lambda_0 \in (0, \infty)$ . Then there exist  $\varepsilon_0 > 0$  and a sequence  $\{\lambda_n\} \subset (\lambda_0/2, 2\lambda_0)$ ,  $\lim_{n \rightarrow \infty} \lambda_n = \lambda_0$  such that  $|S_+(\lambda_n) - S_+(\lambda_0)| \geq \varepsilon_0$  for  $n \in \mathbb{N}$ , that is

$$|u'_{c_{\lambda_n}}(c_{\lambda_n}) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \geq \varepsilon_0 \quad \text{for } n \in \mathbb{N}. \quad (4.4)$$

We claim that there exists  $\nu > 0$  such that

$$|u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_n}}(c_{\lambda_n})| < \frac{\varepsilon_0}{2} \quad \text{for } t \in [c_{\lambda_n} - \nu, c_{\lambda_n}], \quad n \in \mathbb{N}. \quad (4.5)$$

If not, without restriction of generality we can assume that there is a sequence  $\{\tau_n\} \subset (0, T)$ ,  $\tau_n < c_{\lambda_n}$ ,  $\lim_{n \rightarrow \infty} (\tau_n - c_{\lambda_n}) = 0$  such that

$$|u'_{c_{\lambda_n}}(\tau_n) - u'_{c_{\lambda_n}}(c_{\lambda_n})| = \frac{\varepsilon_0}{2} \quad \text{for } n \in \mathbb{N}. \quad (4.6)$$

If  $u'_{c_{\lambda_n}}(\tau_n) \leq 0$ , then  $r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \leq 0$  for  $t \in [\tau_n, c_{\lambda_n}]$  and integrating the inequality

$$(r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t))' r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \leq -Qg(u_{c_{\lambda_n}}(t))r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \quad (4.7)$$

from  $\tau_n$  to  $c_{\lambda_n}$  we get

$$\begin{aligned} & (0 \leq) (r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 - (r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \\ & \leq -2Q \int_{u_{c_{\lambda_n}}(\tau_n)}^0 g(s)r(s) ds = 2Q \int_0^{u_{c_{\lambda_n}}(\tau_n)} g(s)r(s) ds. \end{aligned} \quad (4.8)$$

If  $u'_{c_{\lambda_n}}(\tau_n) > 0$ , then there exists  $\xi_n \in (\tau_n, c_{\lambda_n})$  such that  $u'_{c_{\lambda_n}}(\xi_n) = 0$  and then integrating the inequality

$$(r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t))' r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) \geq -Qg(u_{c_{\lambda_n}}(t))r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t)$$

from  $\tau_n$  to  $\xi_n$  and inequality (4.7) from  $\xi_n$  to  $c_{\lambda_n}$ , we have

$$(r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \leq 2Q \int_{u_{c_{\lambda_n}}(\tau_n)}^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s) ds \quad (4.9)$$

and

$$(r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 \leq 2Q \int_0^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s) ds. \quad (4.10)$$

Let  $\mathbb{N}_+$  be the set of all  $n \in \mathbb{N}$  such that  $u'_{c_{\lambda_n}}(\tau_n) > 0$ . Assume  $\mathbb{N}_+$  is infinite. Using (4.3) and the equalities  $\lim_{n \rightarrow \infty}(\xi_n - c_{\lambda_n}) = 0$ ,  $\lim_{n \rightarrow \infty}(\xi_n - \tau_n) = 0$ , we have

$$\lim_{n \in \mathbb{N}_+, n \rightarrow \infty} (u_{c_{\lambda_n}}(\xi_n) - u_{c_{\lambda_n}}(\tau_n)) = 0,$$

$$\lim_{n \in \mathbb{N}_+, n \rightarrow \infty} u_{c_{\lambda_n}}(\xi_n) = \lim_{n \in \mathbb{N}_+, n \rightarrow \infty} (u_{c_{\lambda_n}}(\xi_n) - u_{c_{\lambda_n}}(c_{\lambda_n})) = 0,$$

and so (4.9), (4.10) and  $r(x) \geq r_0 > 0$  for  $x \in \mathbb{R}$  yield

$$\lim_{n \in \mathbb{N}_+, n \rightarrow \infty} u'_{c_{\lambda_n}}(\tau_n) = \lim_{n \in \mathbb{N}_+, n \rightarrow \infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = 0,$$

contrary to (4.6). Hence  $\mathbb{N}_+$  is finite and there is no loss of generality in assuming  $u'_{c_{\lambda_n}}(\tau_n) \leq 0$  for  $n \in \mathbb{N}$  and then (see (4.8))

$$(0 \leq) (r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 - (r(u_{c_{\lambda_n}}(\tau_n))u'_{c_{\lambda_n}}(\tau_n))^2 \leq 2Q \int_0^{u_{c_{\lambda_n}}(\tau_n)} g(s)r(s) ds \quad (4.11)$$

for  $n \in \mathbb{N}$ . From Lemma 1.2 in [14] (with  $\mu = 1$ ) it follows that

$$u'_{c_{\lambda_n}}(c_{\lambda_n}) \leq -\frac{2K}{Vc_{\lambda_0/2}} \quad \text{for } n \in \mathbb{N}, \quad (4.12)$$

where

$$V = \max \left\{ r(x) : 0 \leq x \leq \max\{u_T(t) : 0 \leq t \leq T\} \right\}, \quad (4.13)$$

$$K = \min \left[ \min \left\{ \int_0^{t/2} s|q(s)|k(s) ds, \int_{t/2}^t (t-s)|q(s)|k(s) ds \right\} : \frac{\lambda_0}{2} \leq t \leq 2\lambda_0 \right]$$

and  $k \in C^0([0, T])$  is a positive function such that

$$0 < k(t) \leq f(t, x)\text{sign } x \quad \text{for } (t, x) \in [0, T] \times ([-\|u_T\|, 0) \cup (0, \|u_T\|]) \quad (4.14)$$

with  $\|u_T\| = \max\{u_T(t) : 0 \leq t \leq T\}$  (for the function  $k$  see Remark 1.3). By (4.3),  $\{u'_{c_{\lambda_n}}(c_{\lambda_n})\}$  and  $\{u'_{c_{\lambda_n}}(\tau_n)\}$  are bounded, and so going if necessary to subsequences, we can assume that they are convergent, say

$$\lim_{n \rightarrow \infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = A, \quad \lim_{n \rightarrow \infty} u'_{c_{\lambda_n}}(\tau_n) = B.$$

By virtue of (4.6), we have

$$|A - B| = \frac{\varepsilon_0}{2}. \quad (4.15)$$

In addition,  $\lim_{n \rightarrow \infty} u_{c_{\lambda_n}}(\tau_n) = \lim_{n \rightarrow \infty} (u_{c_{\lambda_n}}(\tau_n) - u_{c_{\lambda_n}}(c_{\lambda_n})) = 0$  since (4.3) holds and  $\lim_{n \rightarrow \infty}(\tau_n - c_{\lambda_n}) = 0$ . Letting  $n \rightarrow \infty$  in (4.11), we get  $0 \leq (r(0))^2(A^2 - B^2) = 0$ . Therefore  $A^2 - B^2 = 0$  and since  $A \leq -2K/(Vc_{\lambda_0/2})$  by (4.12) and  $B \leq 0$ , we have  $A = B$ , contrary to (4.15). We have proved that (4.5) holds. Let

$$|u'_{c_{\lambda_0}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \leq \frac{\varepsilon_0}{4} \quad \text{for } t \in [c_{\lambda_0} - \gamma, c_{\lambda_0}], \quad (4.16)$$

where  $\gamma$  is a positive constant. Set  $\kappa = \min\{\nu, \gamma\}$ . Using (4.4) and (4.5) we have

$$|u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0})| \geq |u'_{c_{\lambda_n}}(c_{\lambda_n}) - u'_{c_{\lambda_0}}(c_{\lambda_0})| - |u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_n}}(c_{\lambda_n})| > \frac{\varepsilon_0}{2}$$

for  $t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}]$  and  $n \in \mathbb{N}$ . Assume that

$$\mathbb{N}^* = \left\{ n : n \in \mathbb{N}, u'_{c_{\lambda_n}}(t) - u'_{c_{\lambda_0}}(c_{\lambda_0}) > \frac{\varepsilon_0}{2} \text{ for } t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}] \right\}$$

is an infinite set (analogously for  $\mathbb{N} \setminus \mathbb{N}^*$  infinite). Then

$$u_{c_{\lambda_n}}(t) = \int_{c_{\lambda_n}}^t u'_{c_{\lambda_n}}(s) ds \leq \left( u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{2} \right) (t - c_{\lambda_n}) \quad (4.17)$$

for  $t \in [c_{\lambda_n} - \kappa, c_{\lambda_n}]$  and  $n \in \mathbb{N}^*$ . On the other hand, (4.16) gives

$$\begin{aligned} & \left( u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{4} \right) (t - c_{\lambda_0}) \leq u_{c_{\lambda_0}}(t) \\ & = \int_{c_{\lambda_0}}^t u'_{c_{\lambda_0}}(s) ds \leq \left( u'_{c_{\lambda_0}}(c_{\lambda_0}) - \frac{\varepsilon_0}{4} \right) (t - c_{\lambda_0}) \end{aligned} \quad (4.18)$$

for  $t \in [c_{\lambda_0} - \kappa, c_{\lambda_0}]$ . Since  $\lim_{n \rightarrow \infty} (c_{\lambda_n} - \kappa) = c_{\lambda_0} - \kappa$ , there exists  $n_0 \in \mathbb{N}$  such that for  $n \in \mathbb{N}^*$ ,  $n \geq n_0$ , we have  $c_{\lambda_n} - \kappa \leq c_{\lambda_0} - \kappa/2$  and then letting  $n \rightarrow \infty$  in (4.17) and using Lemmas 4.1 and 4.2,

$$u_{c_{\lambda_0}}(t) \leq \left( u'_{c_{\lambda_0}}(c_{\lambda_0}) + \frac{\varepsilon_0}{2} \right) (t - c_{\lambda_0}), \quad t \in \left[ c_{\lambda_0} - \frac{\kappa}{2}, c_{\lambda_0} \right],$$

contrary to (4.18). We have proved that  $S_+$  is continuous on  $(0, \infty)$ .

Let  $\{\lambda_n\} \subset (0, \infty)$ ,  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ . Then  $\lim_{n \rightarrow \infty} c_{\lambda_n} = 0$  by Lemma 4.1, and there exists  $\{\xi_n\}$ ,  $0 < \xi_n < c_{\lambda_n}$ , such that  $u'_{c_{\lambda_n}}(\xi_n) = 0$  and  $r(u_{c_{\lambda_n}}(t))u'_{c_{\lambda_n}}(t) < 0$  on  $(\xi_n, c_{\lambda_n}]$  and  $n \in \mathbb{N}$ . Integrating (4.7) from  $\xi_n$  to  $c_{\lambda_n}$  we get

$$(r(0)u'_{c_{\lambda_n}}(c_{\lambda_n}))^2 \leq 2Q \int_0^{u_{c_{\lambda_n}}(\xi_n)} g(s)r(s) ds \quad \text{for } n \in \mathbb{N}. \quad (4.19)$$

By Lemma 1.2 in [14] (with  $\mu = 1$ ),  $u_{c_{\lambda_n}}(t) \leq L_n$  for  $t \in [0, c_{\lambda_n}]$  where  $L_n > 0$  is an arbitrary constant satisfying the inequality

$$1 \leq \frac{2 \left( \int_0^{L_n} r(s) ds \right)^2}{(c_{\lambda_n})^2 Q \int_0^{L_n} g(s)r(s) ds}.$$

From the last inequality we see that  $L_n$  can be chosen such that  $\lim_{n \rightarrow \infty} L_n = 0$  and then (4.19) yields  $\lim_{n \rightarrow \infty} u'_{c_{\lambda_n}}(c_{\lambda_n}) = 0$ . Hence  $\lim_{\lambda \rightarrow \infty} S_+(\lambda) = 0$ .

Let  $\{\lambda_n\} \subset (0, \infty)$ ,  $\lim_{n \rightarrow \infty} \lambda_n = 0$ . Then  $\lim_{n \rightarrow \infty} c_{\lambda_n} = T$  by Lemma 4.1, and from Lemma 1.2 in [14] (with  $\mu = 1$ ) we deduce that for each  $n \in \mathbb{N}$  such that  $c_{\lambda_n} \geq T/2$ ,

$$u'_{c_{\lambda_n}}(c_{\lambda_n}) \leq -\frac{2K_1}{VT}, \quad n \in \mathbb{N},$$

where  $V$  is given by (4.13) and

$$K_1 = \min \left[ \min \left\{ \int_0^{t/2} s|q(s)|k(s) ds, \int_{t/2}^t (t-s)|q(s)|k(s) ds \right\} : \frac{T}{2} \leq t \leq T \right]$$

with  $k \in C^0([0, T])$  satisfying (4.14). Hence  $\limsup_{\lambda \rightarrow 0^+} S_+(\lambda) \leq -2K_1/(VT)$ .  $\square$

Define the functions  $S_- : (0, \infty) \rightarrow (0, \infty)$ ,  $Z_+ : (0, \infty) \rightarrow (0, \infty)$  and  $Z_- : (0, \infty) \rightarrow (-\infty, 0)$  by the formulas

$$\begin{aligned} S_-(\lambda) &= \overline{u}'_{\alpha_\lambda}(\alpha_\lambda), \\ Z_+(\lambda) &= v'_{c_\lambda}(\alpha_\lambda), \\ Z_-(\lambda) &= \overline{v}'_{\alpha_\lambda}(c_\lambda). \end{aligned}$$

We observe that  $c_\lambda$  (resp.  $\alpha_\lambda$ ) is the (unique) solution of the equation  $\lambda\Lambda_+(c) + \Phi_-(c) = 0$  (resp.  $\lambda\Phi_+(\alpha) + \Lambda_-(\alpha) = 0$ ). From the properties of the functions  $\Lambda_+$ ,  $\Lambda_-$ ,  $\Phi_+$ ,  $\Phi_-$ , using Remarks 2.1 and 2.2 and applying procedures analogical to those in the proofs of Lemmas 4.1–4.3, we can show properties of the functions  $S_-$ ,  $Z_+$  and  $Z_-$  which are given in the following lemma.

**Lemma 4.4.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then the functions  $S_-$ ,  $Z_+$  and  $Z_-$  are continuous on  $(0, \infty)$  and

$$\begin{aligned} \liminf_{\lambda \rightarrow 0^+} S_-(\lambda) &> 0, \quad \lim_{\lambda \rightarrow \infty} S_-(\lambda) = 0, \\ \lim_{\lambda \rightarrow 0^+} Z_+(\lambda) &= 0, \quad \liminf_{\lambda \rightarrow \infty} Z_+(\lambda) > 0, \\ \lim_{\lambda \rightarrow 0^+} Z_-(\lambda) &= 0, \quad \limsup_{\lambda \rightarrow \infty} Z_-(\lambda) < 0. \end{aligned}$$

**Theorem 4.5.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then problem (1.1)-(1.3) has at least two solutions.

**Proof.** Define the function  $k, p : (0, \infty) \rightarrow \mathbb{R}$  by

$$k(\lambda) = S_+(\lambda) - Z_-(\lambda), \quad p(\lambda) = S_-(\lambda) - Z_+(\lambda).$$

By Lemmas 4.3 and 4.4, the functions  $k$  and  $p$  are continuous on  $(0, \infty)$  and

$$\begin{aligned} \limsup_{\lambda \rightarrow 0^+} k(\lambda) &< 0, \quad \liminf_{\lambda \rightarrow \infty} k(\lambda) > 0, \\ \liminf_{\lambda \rightarrow 0^+} p(\lambda) &> 0, \quad \limsup_{\lambda \rightarrow \infty} p(\lambda) < 0. \end{aligned}$$

Hence there exist  $\lambda_1, \lambda_2 \in (0, \infty)$  such that  $k(\lambda_1) = 0$  and  $p(\lambda_2) = 0$ , that is  $S_+(\lambda_1) = Z_-(\lambda_1)$  and  $S_-(\lambda_2) = Z_+(\lambda_2)$ . Then the functions

$$x_1(t) = \begin{cases} u_{c_1}(t) & \text{for } t \in [0, c_1] \\ \overline{v}_{c_1}(t) & \text{for } t \in (c_1, T] \end{cases}$$

and

$$x_2(t) = \begin{cases} \overline{u}_{\alpha_1}(t) & \text{for } t \in [0, \alpha_1] \\ v_{\alpha_1}(t) & \text{for } t \in (\alpha_1, T] \end{cases}$$

are solutions of problem (1.1)-(1.3), where  $c_1$  (resp.  $\alpha_1$ ) is the (unique) solution of the equation  $\lambda_1\Lambda_+(c) + \Phi_-(c) = 0$  (resp.  $\lambda_2\Phi_+(\alpha) + \Lambda_-(\alpha) = 0$ ). Clearly,  $x_1 \neq x_2$ .  $\square$



### 5. Exceptional $n$ -sign-changing $w$ -solutions of problem (1.1), (1.2), (5.1)

Let  $c \in (0, T]$ . In this Section we will use the following conditions

$$\max\{x(t) : 0 \leq t \leq T\} = -\min\{x(t) : 0 \leq t \leq T\}, \quad (5.1)$$

$$x(0) = 0, \quad x(c) = 0 \quad (5.2)$$

and

$$\max\{x(t) : 0 \leq t \leq c\} = -\min\{x(t) : 0 \leq t \leq c\}. \quad (5.3)$$

We note that (5.1) is (1.3) with  $\lambda = 1$ .

**Definition 5.1.** Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . We say that  $x$  is an  *$n$ -sign-changing  $w$ -solution of problem (1.1), (5.2), (5.3)* if  $x$  has precisely  $n - 1$  zeros  $t_1 < t_2 < \dots < t_{n-1}$  in  $(0, c)$ ,  $x \in C^0([0, c]) \cap C^1([0, c] \setminus \{t_1, t_2, \dots, t_{n-1}\})$ , there exist finite  $\lim_{t \rightarrow t_i^-} x'(t)$ ,  $\lim_{t \rightarrow t_i^+} x'(t)$  for  $i = 1, 2, \dots, n - 1$ ,  $r(x)x' \in C^1([0, c] \setminus \{t_1, t_2, \dots, t_{n-1}\})$ ,  $x$  satisfies (5.2), equality (1.1) holds on  $(0, c) \setminus \{t_1, t_2, \dots, t_{n-1}\}$  and finally

$$\max\{x(t) : t_i \leq t \leq t_{i+2}\} \min\{x(t) : t_i \leq t \leq t_{i+2}\} < 0$$

for  $i = 0, 1, \dots, n - 2$  with  $t_0 = 0$  and  $t_n = c$ .

If, in addition,

$$\max\{|x(t)| : 0 \leq t \leq t_1\} = \max\{|x(t)| : t_j \leq t \leq t_{j+1}\}$$

for  $j = 1, 2, \dots, n - 1$ , we say that  $x$  is an *exceptional  $n$ -sign-changing  $w$ -solution of problem (1.1), (5.2), (5.3)*. In case of  $c = T$ ,  $x$  is called an *exceptional  $n$ -sign-changing  $w$ -solution of problem (1.1), (1.2), (5.1)*.

**Remark 5.2.** We observe, that the notion of the  $w$ -solution of problem (1.1)-(1.3) with  $\lambda = 1$  stated in Section 1 corresponds to the notion of exceptional 2-sign-changing  $w$ -solution of problem (1.1), (1.2), (5.1).

Before we give existence results for exceptional  $n$ -sign-changing  $w$ -solutions of problem (1.1), (1.2), (5.1), we will define a function  $\Delta_+$  whose properties are important in our next considerations.

Let assumptions  $(H_1) - (H_4)$  be satisfied. Then, by Theorem 3.2 (with  $\mu = 1$ ) and its proof, for each  $c \in (0, T]$  there exists the unique  $w$ -solution of problem (1.1), (5.2), (5.3), which is positive in the right neighbourhood of  $t = 0$ . This solution we will denote throughout this Section by  $w_c$ . Using  $w_c$  we define the function  $\Delta_+ : (0, T] \rightarrow (0, \infty)$  by

$$\Delta_+(c) = \max\{w_c(t) : 0 \leq t \leq c\}.$$

**Lemma 5.3.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then  $\Delta_+$  is continuous and increasing on  $(0, T]$  and

$$\lim_{c \rightarrow 0^+} \Delta_+(c) = 0. \quad (5.4)$$

**Proof.** To prove that  $\Delta_+$  is increasing on  $(0, T]$  we assume, on the contrary, that  $\Delta_+(a) \geq \Delta_+(b)$  for some  $0 < a < b \leq T$ . Let  $w_a(t_a) = 0$  and  $w_b(t_b) = 0$  with unique  $t_a \in (0, a)$  and  $t_b \in (0, b)$ . From Lemma 2.9 and our assumption  $\Delta_+(a) \geq \Delta_+(b)$  we deduce that  $t_a \geq t_b$ . We claim that

$$w_b(t) < w_a(t) \quad \text{for } t \in (t_a, a]. \quad (5.5)$$

If not, since  $0 = w_a(t_a) \geq w_b(t_a)$  and  $0 = w_a(a) > w_b(a)$  we have either  $w_a(\xi) = w_b(\xi)$  for some  $\xi \in (t_a, a)$  and  $w_b(t) \leq w_a(t)$  for  $t \in [t_a, a]$  or there exist  $t_a \leq \nu < \tau < a$  such that  $w_a(\nu) = w_b(\nu)$ ,  $w_a(\tau) = w_b(\tau)$  and  $w_b(t) > w_a(t)$  for  $t \in (\nu, \tau)$ . In the first case  $w'_a(\xi) = w'_b(\xi)$  and  $w_a = w_b$  in a neighbourhood of  $t = \xi$  by  $(H_4)$ , and then by repeated application of  $(H_4)$  we get  $w_a = w_b$  on  $(t_a, a)$ , which is impossible. In the second case, we have  $r(w_a(\nu))w'_a(\nu) \leq r(w_b(\nu))w'_b(\nu)$  and  $f(t, w_a(t)) \geq f(t, w_b(t))$  for  $t \in (\nu, \tau]$ . Hence

$$\left( \int_{w_a(t)}^{w_b(t)} r(s) ds \right)'' = q(t)(f(t, w_b(t)) - f(t, w_a(t))) \geq 0, \quad t \in (\nu, \tau],$$

and so  $\left( \int_{w_a(t)}^{w_b(t)} r(s) ds \right)'$  is nondecreasing on  $[\nu, \tau]$  and then the equalities  $w_a(\nu) = w_b(\nu)$ ,  $w_a(\tau) = w_b(\tau)$  imply  $\int_{w_a(t)}^{w_b(t)} r(s) ds = 0$  for  $t \in [\nu, \tau]$ , contrary to  $w_b > w_a$  on  $(\nu, \tau)$ . Now (5.5) yields  $\min\{w_b(t) : 0 \leq t \leq b\} < \min\{w_a(t) : 0 \leq t \leq a\}$ , hence  $\Delta_+(b) > \Delta_+(a)$ , contrary to our assumption  $\Delta_+(a) \geq \Delta_+(b)$ . We have proved that  $\Delta_+$  is increasing on  $(0, T]$ .

Suppose that  $\Delta_+$  is discontinuous on the right at a point  $c_0 \in (0, T)$ , i.e., there is a decreasing sequence  $\{c_n\} \subset (c_0, T)$  such that  $\lim_{n \rightarrow \infty} c_n = c_0$  and

$$\lim_{n \rightarrow \infty} \Delta_+(c_n) = \mu > \Delta_+(c_0). \quad (5.6)$$

Let  $w_{c_n}(t_n) = 0$  for the (unique)  $t_n \in (0, c_n)$ ,  $n \in \mathbb{N} \cup \{0\}$ . Since  $\mu < \Delta_+(c_{n+1}) < \Delta_+(c_n)$  for  $n \in \mathbb{N}$ , Lemma 2.9 shows that  $t_0 < t_{n+1} < t_n$  for  $n \in \mathbb{N}$ . There is no loss of generality in assuming  $t_1 < c_0$ . Moreover,  $\Delta_+(c_n) = \Delta_+(t_n)$  for  $n \in \mathbb{N} \cup \{0\}$  and from  $\Delta_+(c_0) < \mu < \Delta_+(t_n)$  and the continuity of  $\Delta_+$  by Lemma 2.6, we see that  $\lim_{n \rightarrow \infty} t_n = t_* > t_0$ . Applying the procedure as in the proof of Lemma 2.6 (now on  $[t_n, c_n]$ ), we get

$$|r(w_{c_n}(t))w'_{c_n}(t)| \leq \sqrt{2Q \int_0^{\Delta_+(t_n)} g(s)r(s) ds} \leq \sqrt{2Q \int_0^{\Delta_+(t_1)} g(s)r(s) ds} \quad (5.7)$$

for  $t \in [t_n, c_n]$ ,  $n \in \mathbb{N}$ , and then

$$\begin{aligned} 0 &\leq \int_{t_n}^t q(s)f(s, w_{c_n}(s)) ds = r(w_{c_n}(t))w'_{c_n}(t) - r(0)w'_{c_n}(t_n) \\ &\leq 2\sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s)r(s) ds} \end{aligned} \quad (5.8)$$

for  $t \in [t_n, c_n]$  and  $n \in \mathbb{N}$ . By (5.7),

$$|w'_{c_n}(t)| \leq S \quad \text{for } t \in [t_n, c_n], \quad n \in \mathbb{N}, \quad (5.9)$$

where

$$S = \frac{1}{r_0} \sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s)r(s) ds}. \quad (5.10)$$

From (5.9),  $0 \geq w_{c_n}(t) \geq -\Lambda_+(t_1)$  for  $t \in [t_n, c_n]$ ,  $n \in \mathbb{N}$ , and the Arzelà–Ascoli theorem, we deduce that there exists a subsequence of  $\{w_{c_n}\}$ , which we denote by  $\{w_{c_n}\}$  again, such that  $\lim_{n \rightarrow \infty} w_{c_n}(t) = w(t)$  locally uniformly on  $(t_*, c_0]$ . Then  $w \in C^0((t_*, c_0])$ ,  $w \leq 0$  on  $(t_*, c_0]$  and  $w(c_0) = 0$  since in the case that  $w(c_0) < 0$  from the relations

$$\frac{w(c_0)}{2} \geq w_{c_n}(c_0) = w_{c_n}(c_0) - w_{c_n}(c_n) = w'_{c_n}(\xi_n)(c_0 - c_n),$$

which are satisfied for sufficiently large  $n$  and where  $\xi_n \in (c_0, c_n)$ , we obtain

$$\lim_{n \rightarrow \infty} w'_{c_n}(\xi_n) \geq \lim_{n \rightarrow \infty} \frac{w(c_0)}{2(c_0 - c_n)} = \infty,$$

contrary to (5.9). We are going to show that  $w(t) < 0$  on  $(t_*, c_0)$  and

$$\lim_{t \rightarrow t_*^+} w(t) = 0. \quad (5.11)$$

By  $(H_3)$  (see Remark 1.3), there exists a positive function  $k \in C^0([0, T])$  such that

$$k(t) \leq f(t, x) \operatorname{sign} x \quad \text{for } (t, x) \in [0, T] \times [-\Delta_+(c_1), 0) \cup (0, \Delta_+(c_1)].$$

Now using our Remark 2.2 and Lemma 1.2 in [14] with  $\mu = 1$ , we get (for  $n \in \mathbb{N}$ )

$$w_{c_n}(t) \leq \begin{cases} H^{-1}\left(-\frac{2K_n(t-t_n)}{c_n-t_n}\right) & \text{for } t \in \left[t_n, \frac{t_n+c_n}{2}\right] \\ H^{-1}\left(-\frac{2K_n(c_n-t)}{c_n-t_n}\right) & \text{for } t \in \left(\frac{t_n+c_n}{2}, c_n\right], \end{cases} \quad (5.12)$$

where

$$K_n = \min \left\{ \int_{t_n}^{(t_n+c_n)/2} (s-t_n)|q(s)|k(s) ds, \int_{(t_n+c_n)/2}^{c_n} (c_n-s)|q(s)|k(s) ds \right\}$$

and  $H^{-1}$  is the inverse to  $H$  given by (2.3). Let  $t_n \leq (3t_* + c_0)/4$  and  $t_n + c_n \leq (t_* + 3c_0)/2$  for  $n \geq n_1$  with an  $n_1 \in \mathbb{N}$ . Then (for  $n \geq n_1$ )

$$\int_{t_n}^{(t_n+c_n)/2} (s-t_n)|q(s)|k(s) ds \geq \int_{(3t_*+c_0)/4}^{(t_*+c_0)/2} \left(s - \frac{3t_*+c_0}{4}\right) |q(s)|k(s) ds,$$

$$\int_{(t_n+c_n)/2}^{c_n} (c_n-s)|q(s)|k(s) ds \geq \int_{(t_*+3c_0)/4}^{c_0} (c_0-s)|q(s)|k(s) ds$$

and from (5.12) it follows that

$$w_{c_n}(t) \leq \begin{cases} H^{-1}\left(-\frac{2K(t-t_n)}{c_n-t_n}\right) & \text{for } t \in \left[t_n, \frac{t_n+c_n}{2}\right] \\ H^{-1}\left(-\frac{2K(c_n-t)}{c_n-t_n}\right) & \text{for } t \in \left(\frac{t_n+c_n}{2}, c_n\right], \end{cases}$$

where

$$K = \min \left\{ \int_{(3t_*+c_0)/4}^{(t_*+c_0)/2} \left(s - \frac{3t_*+c_0}{4}\right) |q(s)|k(s) ds, \int_{(t_*+3c_0)/4}^{c_0} (c_0-s)|q(s)|k(s) ds \right\}.$$

Consequently,

$$w(t) = \lim_{n \rightarrow \infty} w_{c_n}(t) \leq \begin{cases} H^{-1}\left(-\frac{2K(t-t_*)}{c_0-t_*}\right) < 0 & \text{for } t \in \left(t_*, \frac{t_*+c_0}{2}\right] \\ H^{-1}\left(-\frac{2K(c_0-t)}{c_0-t_*}\right) < 0 & \text{for } t \in \left(\frac{t_*+c_0}{2}, c_0\right), \end{cases}$$

and we see that  $w < 0$  on  $(t_*, c_0)$ . If (5.11) is not true, then there exist  $\delta < 0$  and a decreasing sequence  $\{\nu_n\} \subset (t_*, c_0)$  such that  $\lim_{n \rightarrow \infty} \nu_n = t_*$  and  $w(\nu_n) \leq \delta$  for  $n \in \mathbb{N}$ . Now let

$$\nu_{n_*} < t_* - \frac{r_0 \delta}{4\sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s)r(s) ds}}$$

for some  $n_* \in \mathbb{N}$ . Then there exists  $n_2 \in \mathbb{N}$  such that  $w_{c_n}(\nu_{n_*}) < \delta/2$  and

$$\frac{\delta}{2} > w_{c_n}(\nu_{n_*}) = w_{c_n}(\nu_{n_*}) - w_{c_n}(t_n) = w'_{c_n}(\varphi_n)(\nu_{n_*} - t_n)$$

for  $n \geq n_2$ , where  $\varphi_n \in (t_n, \nu_{n_*})$ . Therefore

$$w'_{c_n}(\varphi_n) < \frac{\delta}{2(\nu_{n_*} - t_n)} < \frac{\delta}{2(\nu_{n_*} - t_*)} < -\frac{2}{r_0} \sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s)r(s) ds}$$

for  $n \geq n_2$ , contrary to (5.9).

Define  $w_* : [t_*, c_0] \rightarrow (-\infty, 0]$  by

$$w_*(t) = \begin{cases} w(t) & \text{for } t \in (t_*, c_0] \\ 0 & \text{for } t = t_*. \end{cases}$$

Then  $w_* \in C^0([t_*, c_0])$ ,  $w_*(t_*) = w_*(c_0) = 0$  and  $w_* < 0$  for  $t \in (t_*, c_0)$ . We are now in a position to show that  $w_*$  is a solution of problem (1.1), (2.2) with  $a = t_*$  and  $b = c_0$ . For this, we define for each  $n \in \mathbb{N}$  the function  $p_n : [t_*, c_0] \rightarrow (-\infty, 0]$  by

$$p_n(t) = \begin{cases} f(t, w_{c_n}(t)) & \text{for } t \in (t_n, c_0] \\ 0 & \text{for } t \in [t_*, t_n]. \end{cases}$$

Since

$$0 \leq \int_{t_*}^{c_0} q(t)p_n(t) dt = \int_{t_n}^{c_0} q(t)f(t, w_{c_n}(t)) dt \leq 2\sqrt{2Q \int_0^{\Lambda_+(t_1)} g(s)r(s) ds}$$

by (5.8) and  $\lim_{n \rightarrow \infty} p_n(t) = f(t, w_*(t))$  for  $t \in (t_*, c_0)$ , Fatou's theorem gives  $q(\cdot)f(\cdot, w_*(\cdot)) \in L_1([t_*, c_0])$ . Fix  $\beta \in (t_*, c_0)$ . Going if necessary to a subsequence, we can assume that  $\{r(w_{c_n}(\beta))w'_{c_n}(\beta)\}$  is convergent,  $\lim_{n \rightarrow \infty} r(w_{c_n}(\beta))w'_{c_n}(\beta) = A$ . Letting  $n \rightarrow \infty$  in

$$\int_{w_{c_n}(\beta)}^{w_{c_n}(t)} r(s) ds = r(w_{c_n}(\beta))w'_{c_n}(\beta)(t - \beta) + \int_{\beta}^t \int_{\beta}^s q(v)f(v, w_{c_n}(v)) dv ds, \quad t \in [t_n, c_n]$$

and using the Lebesgue dominated theorem, we get

$$\int_{w_*(\beta)}^{w_*(t)} r(s) ds = A(t - \beta) + \int_{\beta}^t \int_{\beta}^s q(v)f(v, w_*(v)) dv ds, \quad t \in [t_*, c_0].$$

Whence

$$w_*(t) = H^{-1}\left(H(w_*(\beta)) + A(t - \beta) + \int_{\beta}^t \int_{\beta}^s q(v)f(v, w_*(v)) dv ds\right), \quad t \in [t_*, c_0]$$

and we see that  $w_* \in C^1([t_*, c_0])$  and  $(r(w_*(t))w'_*(t))' = q(t)f(t, w_*(t))$  for  $t \in (t_*, c_0)$ . Therefore  $w_*$  is a solution of problem (1.1), (2.2) with  $a = t_*$  and  $b = c_0$ . Finally, applying Remarks 2.1 and 2.2 to Lemma 2.9, we have  $w_{c_0}(t) < w_*(t)$  for  $t \in [t_*, c_0)$ , and consequently

$$-\Delta_+(c_0) = \min\{w_{c_0}(t) : t_0 \leq t \leq c_0\} < \min\{w_*(t) : t_* \leq t \leq c_0\} = -\mu,$$

contrary to (5.6). Hence  $\Delta_+$  is continuous on the right on  $(0, T)$ .

The continuity of  $\Delta_+$  on the left on  $(0, T]$  can be proved similarly.  $\square$

Now, define the ‘dual’ function  $\Delta_- : (0, T] \rightarrow (0, \infty)$  to  $\Delta_+$  by the formula

$$\Delta_-(c) = \min\{\overline{w}_c(t) : 0 \leq t \leq c\},$$

where  $\overline{w}_c$  is the unique solution of problem (1.1), (5.2), (5.3) such that  $\overline{w}_c < 0$  in the right neighbourhood of  $t = 0$ .

**Lemma 5.4.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then  $\Delta_-$  is continuous decreasing on  $(0, T]$  and

$$\lim_{c \rightarrow 0^+} \Delta_-(c) = 0.$$

**Proof.** Since the proof of the lemma is similar to that of Lemma 5.3, we will omit it.  $\square$

**Lemma 5.5.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then there exist exactly two exceptional 3-sign-changing  $w$ -solutions of problem (1.1), (1.2), (5.1).

**Proof.** By Lemmas 2.7, 2.10 and 5.3,  $\Phi_+$  is continuous decreasing on  $[0, T)$ ,  $\Delta_+$  is continuous increasing on  $(0, T]$  and  $\lim_{c \rightarrow T^-} \Phi_+(c) = \lim_{c \rightarrow 0^+} \Delta_+(c) = 0$ . Set  $p_+(c) = \Delta_+(c) - \Phi_+(c)$  for  $c \in (0, T)$ . Then  $p_+ \in C^0((0, T))$ ,  $\lim_{c \rightarrow 0^+} p_+(c) = -\Phi_+(0) < 0$ ,  $\lim_{c \rightarrow T^-} p_+(c) = \Delta_+(T) > 0$  and since  $p_+$  is increasing on  $(0, T)$ , there is the unique  $c_+ \in (0, T)$  such that  $p_+(c_+) = 0$ . Hence the function

$$x_+(t) = \begin{cases} w_{c_+}(t) & \text{for } t \in [0, c_+] \\ v_{c_+}(t) & \text{for } t \in (c_+, T] \end{cases}$$

is an exceptional 3-sign-changing  $w$ -solution of problem (1.1), (1.2), (5.1), which is positive in the right neighbourhood of  $t = 0$ . Assume that  $\bar{x}_+$  is an additional exceptional 3-sign-changing  $w$ -solution of problem (1.1), (1.2), (5.1) having positive values in the right neighbourhood of  $t = 0$  and let  $\bar{x}_+(t_j) = 0$ ,  $j = 1, 2$ , with  $0 < t_1 < t_2 < T$ . Since  $\bar{x}_+ \neq x_+$ , it is necessary  $t_2 \neq c_+$ , say  $t_2 > c_+$ . Then  $\Delta_+(t_2) - \Phi_+(t_2) > 0$  and from this inequality we deduce that

$$\max\{\bar{x}_+(t) : 0 \leq t \leq t_2\} = \Delta_+(t_2) > \Phi_+(t_2) = \max\{\bar{x}_+(t) : t_2 \leq t \leq T\},$$

contrary to the definition of an exceptional 3-sign-changing  $w$ -solution of problem (1.1), (1.2), (5.1).

By Lemmas 2.8, 2.10 and 5.4,  $\Phi_-$  is continuous increasing on  $[0, T)$ ,  $\Delta_-$  is continuous decreasing on  $(0, T]$  and  $\lim_{c \rightarrow T^-} \Phi_-(c) = \lim_{c \rightarrow 0^+} \Delta_-(c) = 0$ . Set  $p_-(c) = \Delta_-(c) - \Phi_-(c)$  for  $c \in (0, T)$ . Then  $p_- \in C^0((0, T))$ ,  $\lim_{c \rightarrow 0^+} p_-(c) = -\Phi_-(0) > 0$ ,  $\lim_{c \rightarrow T^-} p_-(c) = \Delta_-(T) < 0$  and since  $p_-$  is decreasing on  $(0, T)$ , there is the unique  $c_- \in (0, T)$  such that  $p_-(c_-) = 0$ . The function

$$x_-(t) = \begin{cases} \bar{w}_{c_-}(t) & \text{for } t \in [0, c_-] \\ \bar{v}_{c_-}(t) & \text{for } t \in (c_-, T] \end{cases}$$

is an exceptional 3-sign-changing  $w$ -solution of problem in the right neighbourhood of  $t = 0$  and  $x_-$  is the unique exceptional 3-sign-changing  $w$ -solution of problem (1.1), (1.2), (5.1) having negative value in the right neighbourhood of  $t = 0$ .  $\square$

**Remark 5.6.** Let assumptions  $(H_1) - (H_4)$  be satisfied and let  $c \in (0, T]$ . Then  $(H_1) - (H_4)$  are satisfied with  $c$  instead of  $T$ , and so there exist exactly two exceptional 3-sign-changing  $w$ -solutions of problem (1.1), (5.2), (5.3) by Lemma 5.5.

**Theorem 5.7.** Let assumptions  $(H_1) - (H_4)$  be satisfied. Then for each  $n \in \mathbb{N}$ ,  $n \geq 2$ , there exist exactly two exceptional  $n$ -sign-changing  $w$ -solutions of problem (1.1), (1.2), (5.1).

**Proof.** We proceed by induction. By Theorem 3.2 (with  $\lambda = 1$ ) and Lemma 5.5, there exist exactly two exceptional  $j$ -sign-changing  $w$ -solutions of problem (1.1), (1.2), (5.1),  $j = 2, 3$ . In addition (see Remark 5.6), for each  $c \in (0, T]$  there exists exactly two exceptional 3-sign-changing  $w$ -solutions of problem (1.1), (5.2), (5.3). Assuming that this result holds for  $n = k \geq 3$ , we will prove it for  $n = k + 1$ . Since the proof is similar to that of Lemma 5.5, we give only its main ideas.

First, we assume that for each  $c \in (0, T]$  there exist exactly two exceptional  $k$ -sign-changing  $w$ -solutions  $w_{kc}$  and  $\overline{w}_{kc}$  of problem (1.1), (5.2), (5.3) such that  $w_{kc}$  is positive and  $\overline{w}_{kc}$  is negative in the right neighbourhood of  $t = 0$ . Now define  $\Delta_+^k : (0, T] \rightarrow (0, \infty)$  and  $\Delta_-^k : (0, T] \rightarrow (0, \infty)$  by

$$\Delta_+^k(c) = \max\{w_{kc}(t) : 0 \leq t \leq c\}$$

and

$$\Delta_-^k(c) = \max\{\overline{w}_{kc}(t) : 0 \leq t \leq c\}.$$

Further, we can proceed as in the proof of Lemmas 5.3 and 5.4 to verify that  $\Delta_+^k$  and  $\Delta_-^k$  are continuous increasing on  $(0, T]$ ,  $\lim_{c \rightarrow 0+} \Delta_+^k(c) = \lim_{c \rightarrow 0+} \Delta_-^k(c) = 0$ . Finally, if  $k$  is an even positive integer, set

$$p_k^+(c) = \Delta_+^k(c) - \Phi_+(c), \quad p_k^-(c) = \Delta_-^k(c) + \Phi_-(c) \quad \text{for } c \in (0, T).$$

Since  $p_k^+$  and  $p_k^-$  are continuous increasing on  $(0, T)$ ,  $\lim_{c \rightarrow 0+} p_k^+(c) = -\Phi_+(0) < 0$ ,  $\lim_{c \rightarrow T-} p_k^+(c) = \Delta_+^k(T) > 0$ ,  $\lim_{c \rightarrow 0+} p_k^-(c) = \Phi_-(0) < 0$  and  $\lim_{c \rightarrow T-} p_k^-(c) = \Delta_-^k(T) > 0$ , there exists the unique solution  $c_+$  (resp.  $c_-$ ) of the equation  $p_k^+(c) = 0$  (resp.  $p_k^-(c) = 0$ ). Then

$$x_+(t) = \begin{cases} w_{kc_+}(t) & \text{for } t \in [0, c_+] \\ v_{c_+}(t) & \text{for } t \in (c_+, T], \end{cases}$$

$$x_-(t) = \begin{cases} \overline{w}_{kc_-}(t) & \text{for } t \in [0, c_-] \\ \overline{v}_{c_-}(t) & \text{for } t \in (c_-, T] \end{cases}$$

are the unique two exceptional  $(k+1)$ -sign-changing  $w$ -solutions of problem (1.1), (1.2), (5.1).

If  $k$  is an odd positive integer, we now set

$$p_k^+(c) = \Delta_+^k(c) + \Phi_-(c), \quad p_k^-(c) = \Delta_-^k(c) - \Phi_+(c) \quad \text{for } c \in (0, T).$$

Then  $p_k^+$  and  $p_k^-$  are continuous increasing on  $(0, T)$ , the equations  $p_k^+(c) = 0$  and  $p_k^-(c) = 0$  have the unique solutions  $c_+$  and  $c_-$ , respectively, and

$$x_+(t) = \begin{cases} w_{kc_+}(t) & \text{for } t \in [0, c_+] \\ \overline{v}_{c_+}(t) & \text{for } t \in (c_+, T], \end{cases}$$

and

$$x_-(t) = \begin{cases} \overline{w}_{kc_-}(t) & \text{for } t \in [0, c_-] \\ v_{c_-}(t) & \text{for } t \in (c_-, T] \end{cases}$$

are the unique two exceptional  $(k+1)$ -sign-changing  $w$ -solutions of problem (1.1), (1.2), (5.1).  $\square$

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